

# NONLINEAR DIFFERENTIAL EQUATIONS ARISING FROM BOOLE NUMBERS AND THEIR APPLICATIONS

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**ABSTRACT.** In this paper, we study nonlinear differential equations satisfied by the generating function of Boole numbers. In addition, we derive some explicit and new interesting identities involving Boole numbers and higher-order Boole numbers arising from our nonlinear differential equations.

## 1. INTRODUCTION

The Boole polynomials,  $Bl_n(x | \lambda)$ , ( $n \geq 0$ ), are given by the generating function

$$(1.1) \quad \frac{1}{1 + (1+t)^\lambda} (1+t)^x = \sum_{n=0}^{\infty} Bl_n(x | \lambda) \frac{t^n}{n!}, \quad (\text{see [5–8, 10, 11, 18]}),$$

where we assume that  $\lambda \neq 0$ .

When  $x = 0$ ,  $Bl_n(x) = Bl_n(0 | \lambda)$ , ( $n \geq 0$ ), are called the Boole numbers. The higher-order Boole polynomials (or Peters polynomials) are also defined by the generating function

$$(1.2) \quad \left( \frac{1}{1 + (1+t)^\lambda} \right)^r (1+t)^x = \sum_{n=0}^{\infty} Bl_n^{(r)}(x | \lambda) \frac{t^n}{n!}, \quad (r \in \mathbb{N}), \quad (\text{see [18]}).$$

The first few Boole and higher-order Boole polynomials are as follows:

$$Bl_0(x | \lambda) = \frac{1}{2}, \quad Bl_1(x | \lambda) = \frac{1}{4}(2x - \lambda), \quad Bl_2(x | \lambda) = \frac{1}{4}(2x(x - \lambda - 1) + \lambda),$$

and

$$Bl_0^{(r)}(x | \lambda) = 2^{-r}, \quad Bl_1^{(r)}(x | \lambda) = 2^{-(r+1)}(2x - \lambda), \\ Bl_2^{(r)}(x | \lambda) = 2^{-(r+2)}(4x(x - 1) + (2 - 4x)\lambda r + r(r - 1)\lambda^2), \dots$$

With the viewpoint of umbral calculus, Boole numbers and polynomials have been studied by several authors (see [1–20]).

Recently, Kim-Kim has studied the following nonlinear differential equations(see [6, 8]):

$$(1.3) \quad \left( \frac{d}{dt} \right)^N F(t) = \frac{(-1)^N}{(1+t)^N} \sum_{j=2}^{N+1} (j-1)!(N-1)!H_{N-1,j-2}F(t)^j, \quad (N \in \mathbb{N}),$$

where

$$H_{N,0} = 1, \quad \text{for all } N,$$

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$$H_{N,1} = H_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N},$$

$$H_{N,j} = \frac{H_{N-1,j-1}}{N} + \frac{H_{N-2,j-1}}{N-1} + \cdots + \frac{H_{0,j-1}}{1}, \quad H_{0,j-1} = 0, \quad (2 \leq j \leq N).$$

From (1.3), they derived some explicit and new identities for the Bernoulli numbers of the second kind and the higher-order Bernoulli numbers of the second kind.

The purpose of this paper is to give some explicit and new identities for the Boole numbers and the higher-order Boole numbers arising from nonlinear differential equations.

## 2. NONLINEAR DIFFERENTIAL EQUATIONS ARISING FROM THE GENERATING FUNCTION OF BOOLE NUMBERS

Let

$$(2.1) \quad F = F(t; \lambda) = \frac{1}{(1+t)^\lambda + 1}.$$

Then, by (2.1), we get

$$(2.2) \quad \begin{aligned} F^{(1)} &= \frac{d}{dt} F(t) \\ &= \left( \frac{1}{(1+t)^\lambda + 1} \right)^2 \frac{(-1)\lambda}{(1+t)} (1+t)^\lambda \\ &= \frac{(-1)\lambda}{1+t} \frac{1}{\left( (1+t)^\lambda + 1 \right)^2} \left( (1+t)^\lambda - 1 + 1 \right) \\ &= \frac{(-1)\lambda}{1+t} (F - F^2), \end{aligned}$$

and

$$(2.3) \quad \begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} \\ &= \frac{(-1)^2 \lambda}{(1+t)^2} (F - F^2) - \frac{\lambda}{1+t} (F^{(1)} - 2FF^{(1)}) \\ &= \frac{(-1)^2 \lambda}{(1+t)^2} (F - F^2) + \frac{(-1)^2 \lambda}{(1+t)^2} (1 - 2F) (F - F^2) \\ &= \frac{(-1)^2 \lambda}{(1+t)^2} \{ (1+\lambda)F - (1+3\lambda)F^2 + 2\lambda F^3 \}. \end{aligned}$$

Continuing this process, we set

$$(2.4) \quad F^{(N)} = \left( \frac{d}{dt} \right)^N F(t) = \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) F^i,$$

where  $N = 0, 1, 2, \dots$

From (2.4), we have

$$(2.5)$$

$$F^{(N+1)}$$

$$\begin{aligned}
&= \frac{d}{dt} F^{(N)} \\
&= \frac{(-1)^{N+1} \lambda N}{(1+t)^{N+1}} \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) F^i + \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) i F^{i-1} F^{(1)} \\
&= \frac{(-1)^{N+1} \lambda N}{(1+t)^{N+1}} \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) F^i + \frac{(-1)^{N+1} \lambda^2}{(1+t)^{N+1}} \sum_{i=1}^{N+1} i a_{i-1}(N; \lambda) F^{i-1} (F - F^2) \\
&= \frac{(-1)^{N+1} \lambda}{(1+t)^{N+1}} \left\{ \sum_{i=1}^{N+1} (N + i\lambda) a_{i-1}(N; \lambda) F^i - \sum_{i=2}^{N+2} (i-1) \lambda a_{i-2}(N; \lambda) F^i \right\} \\
&= \frac{(-1)^{N+1} \lambda}{(1+t)^{N+1}} \left\{ (N + \lambda) a_0(N; \lambda) F - (N+1) \lambda a_N(N; \lambda) F^{N+2} \right. \\
&\quad \left. + \sum_{i=2}^{N+1} ((N + i\lambda) a_{i-1}(N; \lambda) - (i-1) \lambda a_{i-2}(N; \lambda) F^i) \right\}.
\end{aligned}$$

On the other hand, replacing  $N$  by  $N+1$  in (2.4), we get

$$(2.6) \quad F^{(N+1)} = \frac{(-1)^{N+1} \lambda}{(1+t)^{N+1}} \sum_{i=1}^{N+2} a_{i-1}(N+1; \lambda) F^i.$$

From (2.5) and (2.6), we can derive the following relations:

$$(2.7) \quad a_0(N+1; \lambda) = (N + \lambda) a_0(N; \lambda),$$

$$(2.8) \quad a_{N+1}(N+1; \lambda) = -(N+1) \lambda a_N(N; \lambda)$$

and

$$(2.9) \quad a_{i-1}(N+1; \lambda) = -(i-1) \lambda a_{i-2}(N; \lambda) + (N + i\lambda) a_{i-1}(N; \lambda),$$

where  $2 \leq i \leq N+1$ .

By (2.1) and (2.4), it is easy to show that

$$(2.10) \quad F = F^{(0)} = \lambda a_0(0; \lambda) F.$$

By comparing the coefficients on both sides of (2.10), we have

$$(2.11) \quad a_0(0; \lambda) = \frac{1}{\lambda}.$$

From (2.2) and (2.4), we note that

$$\begin{aligned}
(2.12) \quad \frac{(-1) \lambda}{1+t} (F - F^2) &= F^{(1)} \\
&= \frac{(-1) \lambda}{1+t} (a_0(1; \lambda) F + a_1(1; \lambda) F^2).
\end{aligned}$$

Thus, by (2.12), we get

$$a_0(1; \lambda) = 1, \text{ and } a_1(1; \lambda) = -1.$$

$$\begin{aligned}
(2.13) \quad a_0(N+1; \lambda) &= (N + \lambda) a_0(N; \lambda) \\
&= (N + \lambda) (N + \lambda - 1) a_0(N-1; \lambda) \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
&= (N + \lambda) (N + \lambda - 1) \cdots (1 + \lambda) a_0 (1; \lambda) \\
&= (N + \lambda) (N + \lambda - 1) \cdots (1 + \lambda) \cdot 1 \\
&= (N + \lambda)_N,
\end{aligned}$$

and

$$\begin{aligned}
(2.14) \quad a_{N+1} (N + 1; \lambda) &= - (N + 1) \lambda a_N (N; \lambda) \\
&= (-1)^2 \lambda^2 (N + 1) N a_{N-1} (N - 1; \lambda) \\
&\vdots \\
&= (-1)^N \lambda^N (N + 1) N \cdots 2 a_1 (1; \lambda) \\
&= (-1)^{N+1} \lambda^N (N + 1)!,
\end{aligned}$$

where

$$(x)_n = x (x - 1) (x - 2) \cdots (x - n + 1), \quad (n \geq 0).$$

From (2.9), we can derive the following equations:

(2.15)

$$\begin{aligned}
&a_1 (N + 1; \lambda) \\
&= -\lambda a_0 (N; \lambda) + (N + 2\lambda) a_1 (N; \lambda) \\
&= -\lambda a_0 (N; \lambda) + (N + 2\lambda) \{-\lambda a_0 (N - 1; \lambda) + ((N - 1) + 2\lambda) a_1 (N - 1; \lambda)\} \\
&= -\lambda (a_0 (N; \lambda) + (N + 2\lambda) a_0 (N - 1; \lambda)) + (N + 2\lambda) (N + 2\lambda - 1) a_1 (N - 1; \lambda) \\
&= -\lambda (a_0 (N; \lambda) + (N + 2\lambda) a_0 (N - 1; \lambda)) \\
&\quad + (N + 2\lambda) (N + 2\lambda - 1) \{-\lambda a_0 (N - 2; \lambda) + (N + 2\lambda - 2) a_1 (N - 2; \lambda)\} \\
&= -\lambda \{a_0 (N; \lambda) + (N + 2\lambda) a_0 (N - 1; \lambda) + (N + 2\lambda) (N + 2\lambda - 1) a_0 (N - 2; \lambda)\} \\
&\quad + (N + 2\lambda) (N + 2\lambda - 1) (N + 2\lambda - 2) a_1 (N - 2; \lambda) \\
&\vdots \\
&= -\lambda \sum_{i=0}^{N-1} (N + 2\lambda)_i a_0 (N - i; \lambda) + (N + 2\lambda)_N a_1 (1; \lambda) \\
&= -\lambda \sum_{i=0}^N (N + 2\lambda)_i a_0 (N - i; \lambda), \\
&a_2 (N + 1; \lambda) \\
&= -2\lambda a_1 (N; \lambda) + (N + 3\lambda) a_2 (N; \lambda) \\
&= -2\lambda a_1 (N; \lambda) + (N + 3\lambda) \{-2\lambda a_1 (N - 1; \lambda) + (N + 3\lambda - 1) a_2 (N - 1; \lambda)\} \\
&= -2\lambda \{a_1 (N; \lambda) + (N + 3\lambda) a_1 (N - 1; \lambda)\} \\
&\quad + (N + 3\lambda) (N + 3\lambda - 1) \{-2\lambda a_1 (N - 2; \lambda) + (N + 3\lambda - 2) a_2 (N - 2; \lambda)\} \\
&= -2\lambda \{a_1 (N; \lambda) + (N + 3\lambda) a_1 (N - 1; \lambda) + (N + 3\lambda) (N + 3\lambda - 1) a_1 (N - 2; \lambda)\} \\
&\quad + (N + 3\lambda) (N + 3\lambda - 1) (N + 3\lambda - 2) a_2 (N - 2; \lambda) \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
&= -2\lambda \sum_{i=0}^{N-2} (N+3\lambda)_i a_1(N-i; \lambda) + (N+3\lambda)_{N-1} a_2(2; \lambda) \\
&= -2\lambda \sum_{i=0}^{N-1} (N+3\lambda)_i a_1(N-i; \lambda),
\end{aligned}$$

and

(2.16)

$$\begin{aligned}
&a_3(N+1; \lambda) \\
&= -3\lambda a_2(N; \lambda) + (N+4\lambda) a_3(N; \lambda) \\
&= -3\lambda \{a_2(N; \lambda) + (N+4\lambda) a_2(N-1; \lambda)\} \\
&\quad + (N+4\lambda)(N+4\lambda-1) \{-3\lambda a_2(N-2; \lambda) + (N+4\lambda-2) a_3(N-2; \lambda)\} \\
&= -3\lambda \{a_2(N; \lambda) + (N+4\lambda) a_2(N-1; \lambda) + (N+4\lambda)(N+4\lambda-1) a_2(N-2; \lambda)\} \\
&\quad + (N+4\lambda)(N+4\lambda-1)(N+4\lambda-2) a_3(N-2; \lambda)
\end{aligned}$$

(2.17)

$\vdots$

(2.18)

$$\begin{aligned}
&= -3\lambda \sum_{i=0}^{N-3} (N+4\lambda)_i a_2(N-i; \lambda) + (N+4\lambda)_{N-2} a_3(3; \lambda) \\
&= -3\lambda \sum_{i=0}^{N-2} (N+4\lambda)_i a_2(N-i; \lambda).
\end{aligned}$$

(2.19)

Proceeding in this way, we get

$$(2.20) \quad a_k(N+1; \lambda) = -k\lambda \sum_{i_1=0}^{N-k+1} (N+(k+1)\lambda)_{i_1} a_{k-1}(N-i_1; \lambda),$$

where  $1 \leq k \leq N$ .

Therefore, we obtain the following theorem.

**Theorem 1.** *We have the following recurrence relations:*

- (i)  $a_0(0; \lambda) = \frac{1}{\lambda}$ ,  $a_0(1; \lambda) = 1$ ,  $a_1(1; \lambda) = -1$ ,
- (ii)  $a_0(N+1; \lambda) = (N+\lambda)_N$ ,  $a_{N+1}(N+1; \lambda) = (-1)^{N+1} \lambda^N (N+1)!$ ,
- (iii)  $a_k(N+1; \lambda) = -k\lambda \sum_{i_1=0}^{N-k+1} (N+(k+1)\lambda)_{i_1} a_{k-1}(N-i_1; \lambda)$ ,

for  $1 \leq k \leq N$ .

Now, we observe that

$$\begin{aligned}
(2.21) \quad a_1(N+1; \lambda) &= -\lambda \sum_{i_1=0}^N (N+2\lambda)_{i_1} a_0(N-i_1; \lambda) \\
&= -\lambda \sum_{i_1=0}^N (N+2\lambda)_{i_1} (N+\lambda-i_1-1)_{N-i_1-1},
\end{aligned}$$

$$(2.22) \quad a_2(N+1; \lambda)$$

$$\begin{aligned}
(2.23) \quad &= -2\lambda \sum_{i_2=0}^{N-1} (N+3\lambda)_{i_2} a_1(N-i_2; \lambda) \\
&= (-1)^2 2! \lambda^2 \sum_{i_2=0}^{N-1} \sum_{i_1=0}^{N-i_2-1} (N+3\lambda)_{i_2} (N+2\lambda-i_2-1)_{i_1} \\
&\quad \times (N+\lambda-i_2-i_1-2)_{N-i_2-i_1-2},
\end{aligned}$$

and

$$\begin{aligned}
(2.24) \quad &a_3(N+1; \lambda) \\
&= -3\lambda \sum_{i_3=0}^{N-2} (N+4\lambda)_{i_3} a_2(N-i_3; \lambda) \\
&= (-1)^3 3! \lambda^3 \sum_{i_3=0}^{N-2} \sum_{i_2=0}^{N-i_3-2} \sum_{i_1=0}^{N-i_3-i_2-2} (N+4\lambda)_{i_3} (N+3\lambda-i_3-1)_{i_2} \\
&\quad \times (N+2\lambda-i_3-i_2-2)_{i_1} \\
&\quad \times (N+\lambda-i_3-i_2-i_1-3)_{N-i_3-i_2-i_1-3}.
\end{aligned}$$

Continuing this process, we have

$$\begin{aligned}
(2.25) \quad &a_j(N+1; \lambda) \\
&= (-1)^j j! \lambda^j \\
&\quad \times \sum_{i_j=0}^{N-j+1} \sum_{i_{j-1}=0}^{N-j+1-i_j} \cdots \sum_{i_1=0}^{N-j+1-i_j-\cdots-i_2} (N+(j+1)\lambda)_{i_j} (N+j\lambda-i_j-1)_{j-1} \\
&\quad \times \cdots \times (N+2\lambda-i_j-\cdots-i_2-(j-1))_{i_1} \\
&\quad \times (N+\lambda-i_j-\cdots-i_1-j)_{N-i_j-\cdots-i_1-j},
\end{aligned}$$

where  $1 \leq j \leq N$ .

From (2.25), we note that the matrix  $(a_i(j; \lambda))_{0 \leq i, j \leq N}$  is given by

$$(2.26) \quad \begin{matrix} & 0 & 1 & 2 & 3 & & N \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \\ N \end{matrix} & \left[ \begin{array}{cccccc} \frac{1}{\lambda} & 1 & (1+\lambda) & (2+\lambda)_2 & \cdots & (N+\lambda-1)_{N-1} \\ & -1 & & & & \\ & & (-1)^2 \lambda 2! & & & \\ & & & (-1)^3 \lambda^2 3! & & \\ & & & & \ddots & \\ & 0 & & & & (-1)^N \lambda^{N-1} N! \end{array} \right] \end{matrix}$$

Therefore, by Theorem 1, (2.4), and (2.25), we obtain the following theorem.

**Theorem 2.** *The nonlinear differential equations*

$$F^{(N)} = \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) F^i, \quad (N \in \mathbb{N}),$$

have a solution  $F = F(t, \lambda) = \frac{1}{(1+t)^\lambda + 1}$ ,

where  $a_0(N; \lambda) = (N + \lambda - 1)_{N-1}$ ,  $a_N(N; \lambda) = (-1)^N \lambda^{N-1} N!$ ,

$$\begin{aligned} a_j(N; \lambda) &= (-1)^j j! \lambda^j \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (N + (j+1)\lambda - 1)_{i_j} \\ &\quad \times (N + j\lambda - \lambda_j - 2)_{i_{j-1}} \cdots (N + 2\lambda - i_j - \cdots - i_2 - j)_{i_1} \\ &\quad \times (N + \lambda - i_j - \cdots - i_1 - j - 1)_{N-i_j-\cdots-i_1-j-1}, \quad (1 \leq j \leq N-1). \end{aligned}$$

Recall that the Boole numbers,  $Bl_k(\lambda)$ , ( $k \geq 0$ ), are given by the generating function

$$(2.27) \quad \frac{1}{(1+t)^\lambda + 1} = \sum_{k=0}^{\infty} Bl_k(\lambda) \frac{t^k}{k!}.$$

Thus, by (2.27), we get

$$\begin{aligned} F^{(N)} &= \left( \frac{d}{dt} \right)^N F(t, \lambda) \\ &= \left( \frac{d}{dt} \right)^N \left( \frac{1}{(1+t)^\lambda + 1} \right) \\ &= \sum_{k=N}^{\infty} Bl_k(\lambda) (k)_N \frac{t^{k-N}}{k!} \\ &= \sum_{k=0}^{\infty} Bl_{k+N}(\lambda) \frac{(k+N)_N}{(k+N)!} t^k \\ &= \sum_{k=0}^{\infty} Bl_{k+N}(\lambda) \frac{t^k}{k!}, \quad (N \in \mathbb{N}). \end{aligned}$$

From (1.2), Theorem 2 and (2.27), we have

$$\begin{aligned} (2.28) \quad &\sum_{k=0}^{\infty} Bl_{k+N}(\lambda) \frac{t^k}{k!} \\ &= F^{(N)} \\ &= \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) \left( \frac{1}{(1+t)^\lambda + 1} \right)^i \\ &= (-1)^N \lambda (1+t)^{-N} \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) \left( \frac{1}{(1+t)^\lambda + 1} \right)^i \end{aligned}$$

$$\begin{aligned}
&= (-1)^N \lambda \left( \sum_{l=0}^{\infty} (-1)^l (N+l-1)_l \frac{t^l}{l!} \right) \left( \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) \sum_{m=0}^{\infty} Bl_m^{(i)}(\lambda) \frac{t^m}{m!} \right) \\
&= (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) \left( \sum_{l=0}^{\infty} (-1)^l (N+l-1)_l \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} Bl_m^{(i)}(\lambda) \frac{t^m}{m!} \right) \\
&= (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) \left( \sum_{k=0}^{\infty} \sum_{l=0}^k \binom{k}{l} (-1)^l (N+l-1)_l Bl_{k-l}^{(i)}(\lambda) \right) \frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} \left\{ (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) \sum_{l=0}^k \binom{k}{l} (-1)^l (N+l-1)_l Bl_{k-l}^{(i)}(\lambda) \right\} \frac{t^k}{k!},
\end{aligned}$$

where  $N \in \mathbb{N}$ .

By comparing the coefficients on both sides of (2.28), we obtain the following theorem.

**Theorem 3.** *For  $N \in \mathbb{N}$  and  $k \in \mathbb{N} \cup \{0\}$ , we have*

$$Bl_{k+N}(\lambda) = (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) \sum_{k=0}^k \binom{k}{l} (-1)^l (N+l-1)_l Bl_{k-l}^{(i)}(\lambda).$$

By replacing  $t$  by  $e^t - 1$  in (1.1), we get

$$\begin{aligned}
(2.29) \quad \frac{1}{2} \left( \frac{2}{e^{\lambda t} + 1} \right) &= \sum_{k=0}^{\infty} Bl_k(\lambda) \frac{1}{k!} (e^t - 1)^k \\
&= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n Bl_k(\lambda) S_2(n, k) \right) \frac{t^n}{n!},
\end{aligned}$$

where  $S_2(n, k)$  are the Stirling numbers of the second kind.

As is well known, Euler numbers are given by the generating function

$$(2.30) \quad \left( \frac{2}{e^t + 1} \right) = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (\text{see [6]}).$$

From (2.29) and (2.30), we have

$$(2.31) \quad 2^{-1} \lambda^n E_n = \sum_{k=0}^n Bl_k(\lambda) S_2(n, k), \quad (n \geq 0).$$

It is well known that the higher-order Euler numbers are also defined by the generating function

$$(2.32) \quad \left( \frac{2}{e^t + 1} \right)^r = \sum_{n=0}^{\infty} E_n^{(r)} \frac{t^n}{n!}, \quad (\text{see [19]}).$$

Now, we observe that

$$\begin{aligned}
(2.33) \quad \left( \frac{1}{e^{\lambda t} + 1} \right)^i &= \left( \frac{1}{(e^t - 1 + 1)^{\lambda} + 1} \right)^i \\
&= \sum_{k=0}^{\infty} Bl_k^{(i)}(\lambda) \frac{1}{k!} (e^t - 1)^k
\end{aligned}$$



$$= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n Bl_k^{(i)}(\lambda) S_2(n, k) \right) \frac{t^n}{n!}.$$

Thus, by (2.32) and (2.33), we get

$$2^{-i} \lambda^n E_n^{(i)} = \sum_{k=0}^n Bl_k^{(i)}(\lambda) S_2(n, k), \quad (n \geq 0, i \in \mathbb{N}).$$

From (1.1) and (2.30), we note that

$$\begin{aligned} (2.34) \quad 2 \sum_{n=0}^{\infty} Bl_n(\lambda) \frac{t^n}{n!} &= \frac{2}{(1+t)^\lambda + 1} \\ &= \frac{2}{e^{\lambda \log(1+t)} + 1} \\ &= \sum_{k=0}^{\infty} E_k \frac{\lambda^k}{k!} (\log(1+t))^k \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n E_k \lambda^k S_1(n, k) \right) \frac{t^n}{n!}, \end{aligned}$$

where  $S_1(n, k)$  are the Stirling numbers of the first kind.

Thus, by (2.34), we get

$$(2.35) \quad Bl_n(\lambda) = \frac{1}{2} \sum_{k=0}^n E_k \lambda^k S_1(n, k), \quad (n \geq 0).$$

By (2.32), we easily get

$$\begin{aligned} (2.36) \quad \left( \frac{2}{(1+t)^\lambda + 1} \right)^i &= \left( \frac{2}{e^{\lambda \log(1+t)} + 1} \right)^i \\ &= \sum_{k=0}^{\infty} E_k^{(i)} \frac{1}{k!} \lambda^k (\log(1+t))^k \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n E_k^{(i)} \lambda^k S_1(n, k) \right) \frac{t^n}{n!}, \quad (i \in \mathbb{N}). \end{aligned}$$

From (1.2) and (2.36), we have

$$(2.37) \quad 2^i Bl_n^{(i)}(\lambda) = \sum_{k=0}^n E_k^{(i)} \lambda^k S_1(n, k), \quad (n \geq 0, i \in \mathbb{N}).$$

Therefore, by Theorem 3, (2.36), and (2.37), we obtain the following theorem.

**Theorem 4.** For  $k \in \mathbb{N} \cup \{0\}$  and  $N \in \mathbb{N}$ , we have

$$\begin{aligned} &\frac{1}{2} \sum_{n=0}^{k+N} E_n \lambda^n S_1(k+N, n) \\ &= (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) \sum_{l=0}^k \binom{k}{l} (-1)^l (N+l-1)_l \sum_{n=0}^{k-l} 2^{-i} E_n^{(i)} \lambda^n S_1(k-l, n). \end{aligned}$$

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